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Symmetric Local Algebras and Small Blocks of Finite Groups

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One of the major problems in modular representation theory is the determination of the numbers $k(B)$ of ordinary and $l(B)$ of modular irreducible characters for the block B of a finite group G with defect group D. Not much is known about the opposite problem of classifying the defect group by the numbers $k(B)$ and $l(B)$. Of course, there is Brauer's result that $k(B)=1$ is equivalent to $D=1$. The next two cases have been solved in a recent paper of J. Brandt [1]. He showed, as a corollary of more general results, that $k(B)=2$, or $l(B)=1$ and $k(B)=3$ imply $|D|=k(B)$. The case $l(B)=2$ and $k(B)=3$ seems to be of a quite different nature. Therefore in this paper we restrict our attention to the case $l(B)=1$ and extend Brandt's result to the case $k(B)\leqslant 4$. It turns out that this theorem follows from some general properties of symmetric local algebras. So we state our result in this more general context.

1. From Blocks to Symmetric Local Algebras

We are going to prove the following theorem.

A. THEOREM. *Let B be a block of the finite group G with defect group D. If $l(B)=1$ and $k(B)\leqslant 4$ then $|D|=k(B)$.*

The possible defect groups are therefore cyclic or a Klein four group. Dade has determined the general structure of a block with cyclic defect group, and Brauer of a block with Klein four defect group. However, we will not use their results.

We first show how this theorem follows from a more general theorem on symmetric local algebras. For us, a symmetric algebra is a pair (A, λ), where

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A is a finite-dimensional algebra with identity over a field F and $\lambda: A \to F$ is an F-linear map such that

(i) $\lambda(xy) = \lambda(yx)$ for all elements $x, y \in A$,

(ii) $\lambda(Ax) = 0$ for an element $x \in A$ if and only if $x = 0$.

It is well known that the group algebra FG of a finite group G over F is symmetric with corresponding linear form

$$\lambda: FG \to F, \qquad \sum_{g \in G} \alpha_g g \mapsto \alpha_1.$$

If e is an arbitrary idempotent in FG, the restriction of λ turns $eFGe$ into a symmetric algebra. We consider blocks of G as two-sided ideals in the group algebra FG over a sufficiently large field F of prime characteristic. We may write $1_B = \sum_{i=1}^{n} e_i$ with pairwise orthogonal primitive idempotents e_i in B. Then we have a decomposition

$$B = \bigoplus_{i,j=1}^{n} e_i B e_j$$

as vector spaces over F. We may define an equivalence relation on $\{e_1, ..., e_n\}$ by calling e_i and e_j equivalent if $e_j = u^{-1} e_i u$ for a unit u in B. Then $l(B)$ is the number of equivalence classes, and $k(B)$ is the dimension of ZB, the center of B. Therefore, if $l(B) = 1$, we can choose units $u_1, ..., u_n \in B$ such that $e_i = u_i^{-1} e_1 u_i$ for $i = 1, ..., n$. Then the map

$$\mathrm{Mat}(n, e_1 B e_1) \to B, \qquad [x_{ij}]_{i,j=1}^{n} \mapsto \sum_{i,j=1}^{n} u_i^{-1} x_{ij} u_j,$$

is easily seen to be an isomorphism of F-algebras; here $\mathrm{Mat}(n, e_1 B e_1)$ denotes the algebra of all $n \times n$-matrices with coefficients in $e_1 B e_1$. In this case, $k(B) = \dim_F ZB = \dim_F Z(e_1 B e_1)$, and the dimension of $e_1 B e_1$ is the only Cartan invariant of B which, by a result of Brauer's, must be the order $|D|$ of D (see also [3]). So our theorem will follow from the fact that $e_1 B e_1$ is commutative. Since $e_1 B e_1 = e_1 FG e_1$ is a symmetric algebra with one-dimensional radical factor $e_1 B e_1 / J(e_1 B e_1)$ it will be enough to prove the following theorem.

B. THEOREM. *Let (A, λ) be a symmetric algebra with* $\dim_F A/JA = 1$ *and* $\dim_F ZA \leqslant 4$. *Then A is commutative.*

From now on, we will not use any other fact from modular representation theory, but our arguments will be purely ring-theoretic.

2. Some Results on Symmetric Local Algebras

We denote by F an arbitrary field. All our algebras will be unitary and of finite F-dimension. For each such algebra A, ZA denotes the center and JA the Jacobson radical of A. For a subset X of A we denote by FX the F-space spanned by X. We will need the commutators $[x, y] = xy - yx$ for elements $x, y \in A$ and the F-space $[X, Y] = F\{[x, y] : x \in X, y \in Y\}$ for subsets X, Y of A. In particular, we have $K(A) = [A, A]$, the space spanned by all commutators in A.

For a symmetric algebra (A, λ), λ can be used to define a non-degenerate symmetric bilinear form on A. Therefore, any subspace X of A has an orthogonal space $X^0 = \{a \in A : \lambda(aX) = 0\}$, and we get

$$\dim_F X + \dim_F X^0 = \dim_F A.$$

The following is well known (see [4]).

C. Lemma. *If (A, λ) is a symmetric algebra then $K(A)^0 = ZA$.*

Proof. For elements $x, y, z \in A$ the equality $\lambda((xy - yx)z) = \lambda(x(yz - zy))$ holds. Therefore z lies in $K(A)^0$ if and only if $\lambda(A(yz - zy)) = 0$ for all elements $y \in A$. But this is equivalent to $yz - zy = 0$ for all elements $y \in A$.

In our next lemma we will need the higher commutators

$$K_1(A) = A, \qquad K_2(A) = K(A), \qquad K_{i+1}(A) = [K_i(A), A]$$

and the higher radical powers $J^iA = (JA)^i$ for natural numbers i and an arbitrary algebra A. We denote by SA the socle of the left A-module A. If (A, λ) is symmetric this is also the socle of the right A-module A, and we have $\dim_F SA = \dim_F A/JA$. In particular, if $\dim_F A/JA = 1$ then $K(A) \cap SA = 0$ since the kernel of λ and therefore $K(A)$ do not contain non-zero ideals of A.

D. Lemma. *For a symmetric algebra (A, λ) with $\dim_F A/JA = 1$ one of the following holds*:

(1) *A is commutative,*

(2) $2 \leqslant \dim_F K(A) \cap ZA \leqslant \dim_F ZA - 2$.

Proof. We may write $A = F1 + JA$. Therefore $K(A) = [JA, JA] \subseteq J^2A$, and by induction we get $K_i(A) \subseteq J^iA$ for all natural numbers $i \geqslant 2$; in particular, $K_n(A) = 0$ for large n. Now assume that A is not commutative. Then $K(A) \cap ZA \subseteq JA \cap ZA = JZA$, and $SA \subseteq JZA$ with $\dim_F SA = 1$ and $K(A) \cap SA = 0$. Therefore $\dim_F K(A) \cap ZA \leqslant \dim_F ZA - 2$. Next we assume

$\dim_F K(A) \cap ZA \leqslant 1$. Then $\dim_F K(A) + ZA \geqslant \dim_F A - 1$, and we may write $A = K(A) + ZA + Fx$ for some element $x \in A$. But then

$$\begin{aligned} K(A) &= [K(A) + Fx, K(A) + Fx] = [K(A), K(A)] + [K(A), Fx] \\ &\subseteq K_3(A) \subseteq K_4(A) \subseteq \cdots \subseteq 0, \end{aligned}$$

and A is commutative.

Lemma D implies Theorem B in the case $\dim_F ZA \leqslant 3$. Therefore, in the following, we only have to deal with the case $\dim_F ZA = 4$. We will need some information on the factors $J^iA/J^{i+1}A$. Therefore the following lemma will be useful.

E. LEMMA. *Let I be an ideal of an algebra A, and let m, n be natural numbers with $m \leqslant n$. Suppose*

$$I^n = F\{x_{i1} \cdots x_{in}: i = 1,..., d\} + I^{n+1}$$

with elements $x_{ij} \in I$. Then

$$I^{n+m} = F\{x_{j1} \cdots x_{jm} x_{i1} \cdots x_{in}: i, j = 1,..., d\} + I^{n+m+1}.$$

In particular, $\dim_F I^{n+m}/I^{n+m+1} \leqslant (\dim_F I^n/I^{n+1})^2$.

Proof. For any element $y \in I^m$ and any number $j \in \{1,..., d\}$,

$$x_{j,m+1} \cdots x_{jn}\, y \in I^n = F\{x_{i1} \cdots x_{in}: i = 1,..., d\} + I^{n+1}.$$

Therefore

$$x_{j1} \cdots x_{jn}\, y \in F\{x_{j1} \cdots x_{jm} x_{i1} \cdots x_{in}: i = 1,..., d\} + I^{n+m+1}.$$

We now prove the crucial property.

F. LEMMA. *Let (A, λ) be a symmetric algebra with* $\dim_F A/JA = 1$ *and* $\dim_F ZA = 4$. *Then* $\dim_F J^2A/J^3A = 1$.

Proof. By Lemma E, any counterexample A must satisfy the following conditions:

$$\dim_F A/JA = 1, \qquad \dim_F JA/J^2A \geqslant 2,$$
$$\dim_F J^2A/J^3A \geqslant 2, \qquad \text{and} \qquad \dim_F J^3A/J^4A \geqslant 1.$$

In particular, A is not commutative. By Lemma D we get $\dim_F K(A) \cap ZA = 2$, and we may write $A = K(A) + ZA + Fx + Fy$ with suitable elements $x, y \in A$. Then

$$K(A) = [K(A) + Fx + Fy,\ K(A) + Fx + Fy] \subseteq K_3(A) + F[x, y],$$

and $\dim_F A/K_3(A) \leqslant 5$. But $K_3(A)$ is contained in J^3A, so from $\dim_F A/J^3A \geqslant 5$ we conclude $J^3A = K_3(A)$. But then J^3A is an ideal of A contained in $K(A)$, and we get the contradiction $J^3A = 0$.

For the proof of the next lemma it will be necessary to define the iterated socles by

$$S_0 A = 0, \qquad S_i A/S_{i-1}A = S(A/S_{i-1}A)$$

for natural numbers i. If (A, λ) is symmetric then we have $(S_i A)^0 = J^i A$ for all i.

G. Lemma. *Let (A, λ) be a symmetric algebra with* $\dim_F A/JA = \dim_F J^n A/J^{n+1}A = 1$ *for some natural number n. Then $J^{n-1}A \subseteq ZA$.*

Proof. Lemma E implies

$$\dim_F J^n A/J^{n+1}A = \cdots = \dim_F J^m A/J^{m+1}A = \dim_F J^m A = 1$$

for some natural number $m \geqslant n$. Since $\dim_F J^i A/J^{i+1}A = \dim_F S_{i+1}A/S_i A$ for all natural numbers i we conclude

$$\dim_F S_{n+1}A/S_n A = \cdots = \dim_F S_{m+1}A/S_m A = \dim_F A/JA = 1.$$

In particular, $A/S_n A$ has exactly one composition series, and $\dim_F J(A/S_n A)/J^2(A/S_n A) \leqslant 1$. Again by Lemma E, $A = F\{1, x, x^2, \ldots\} + S_n A$ for some element $x \in JA$ and

$$K(A) = [F\{x, x^2, \ldots\} + S_n A, F\{x, x^2, \ldots\} + S_n A] \subseteq S_{n-1}A.$$

Therefore $J^{n-1}A = (S_{n-1}A)^0 \subseteq K(A)^0 = ZA$.

Theorem B is now an immediate consequence of F and G.

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